COMBINATORICA 4 (2—3) (1984) 133—140

AN ERDŐS—KO—RADO THEOREM FOR THE SUBCUBES OF A CUBE

KONRAD ENGEL

Received 28 March 1983

Let P be that partially ordered set whose elements are vectors $\mathbf{x}=(x_1, \ldots, x_n)$ with $x_i \in \{0, \ldots, k\}$ $(i=1, \ldots, n)$ and in which the order is given by $\mathbf{x} \leqq \mathbf{y}$ iff $x_i=y_i$ or $x_i=0$ for all i. Let $N_i(P) = \{\mathbf{x} \in P: |\{j: x_j \neq 0\}|=i\}$. A subset F of P is called an Erdős—Ko—Rado family, if for all $\mathbf{x}, \mathbf{y} \in F$ it holds $\mathbf{x} \not< \mathbf{y}$, $\mathbf{x} \not> \mathbf{y}$, and there exists a $\mathbf{z} \in N_1(P)$ such that $\mathbf{z} \leqq \mathbf{x}$ and $\mathbf{z} \leqq \mathbf{y}$. Let $\mathscr{F}$ be the set of all vectors $\mathbf{f}=(f_0, \ldots, f_n)$ for which there is an Erdős—Ko—Rado family F in P such that $|N_i(P) \cap F| = f_i$ $(i=0, \ldots, n)$ and let $\langle \mathscr{F} \rangle$ be its convex closure in the $(n+1)$-dimensional Euclidean space. It is proved that for $k \geqq 2$ $(0, \ldots, 0)$ and $\Big(0, \ldots, 0, \underbrace{\binom{n-1}{i-1} k^{i-1}}_{i\text{-component}}, 0, \ldots, 0\Big)$ $(i=1, \ldots, n)$ are the vertices of $\langle \mathscr{F} \rangle$.

1. Introduction

Let P be that partially ordered set which is the product of n factors

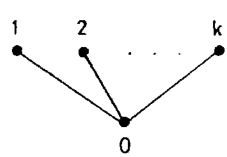

Fig. 1

i.e. the elements of P are vectors $\mathbf{x}=(x_1, \ldots, x_n)$ with $x_i \in \{0, \ldots, k\}$ and the order is given by $\mathbf{x} \leqq \mathbf{y}$ iff $x_i=y_i$ or $x_i=0$ for all $i \in \{1, \ldots, n\}$. If $k \geqq 2$, these posets appear, for instance, if one defines the cube to be the set of all vectors $\mathbf{u}=(u_1, \ldots, u_n)$ with $u_i \in \{1, \ldots, k\}$, a subcube to be a subset of the cube, where certain components of its elements are fixed and the other are free, and the order between the subcubes to be dual to that order which is given by set-inclusion (see [1], [7], or [8]). For $\mathbf{x}=(x_1, \ldots, x_n) \in P$ let $Z(\mathbf{x})=\{i: x_i=0\}$ be the zero set of $\mathbf{x}$, $r(\mathbf{x})=n-|Z(\mathbf{x})|$ be

AMS subject classification (1980): 05 A 99, 06 A 10 the rank of $\mathbf{x}$ and $N_i(P)=\{\mathbf{x}\in P:\ r(\mathbf{x})=i\}$ be the i-th level of P. In honour of the authors of [2] a subset F of P is called an Erdős—Ko—Rado family (E. K. R. family) iff for all $\mathbf{x}, \mathbf{y}\in F$ it holds

$$\text{(1)}\qquad \mathbf{x} \nless \mathbf{y}, \quad \mathbf{x} \ngtr \mathbf{y},$$

$$\text{(2)}\qquad \mathbf{z} \leqq \mathbf{x} \quad \text{and} \quad \mathbf{z} \leqq \mathbf{y} \quad \text{for some} \quad \mathbf{z}\in N_1(P).$$

Obviously, (2) is satisfied iff for any $\mathbf{x}, \mathbf{y}\in F$ there is an index $i\in\{1, \dots, n\}$ such that $x_i=y_i\in\{1, \dots, k\}$. A vector $\mathbf{f}=(f_0, \dots, f_n)$ is called an E. K. R. vector iff there is an E. K. R. family F in P for which $|N_i(P)\cap F|=f_i$ holds for all i. Let $\mathcal{F}$ be the set of all E. K. R. vectors and $\langle\mathcal{F}\rangle$ be the convex closure of $\mathcal{F}$ in the $(n+1)$-dimensional Euclidean space. From linear programming we know: If one wants to maximize (or minimize) the linear function $\sum_{i=0}^{n} c_i f_i$ over all vectors $\mathbf{f}\in\mathcal{F}$, one must only maximize (or minimize) this function over all vertices of the polyhedron $\langle\mathcal{F}\rangle$. Therefore it is interesting to know all vertices of $\langle\mathcal{F}\rangle$. In the case $k=1$, Peter L. Erdős, P. Frankl, and G. O. H. Katona [3] determined these vertices.

Theorem 1. *If* $k=1$, *the vertices of* $\langle\mathcal{F}\rangle$ *are*

$$(0, \dots, 0),$$

$$\Big(0, \dots, 0, \underbrace{\binom{n-1}{i-1}}_{i\text{-component}}, 0, \dots, 0\Big), \quad 1 \leqq i \leqq \frac{n}{2},$$

$$\Big(0, \dots, 0, \underbrace{\binom{n}{j}}_{j\text{-component}}, 0, \dots, 0\Big), \quad \frac{n}{2} < j \leqq n,$$

$$\Big(0, \dots, 0, \underbrace{\binom{n-1}{i-1}}_{i\text{-component}}, 0, \dots, 0, \underbrace{\binom{n-1}{j}}_{j\text{-component}}, 0, \dots, 0\Big), \quad 1 \leqq i \leqq \frac{n}{2}, \quad i+j > n. \quad \blacksquare$$

Our main result is the following

Theorem 2. *If* $k>1$, *the vertices of* $\langle\mathcal{F}\rangle$ *are*

$$\mathbf{v}^0 = (0, \dots, 0),$$

$$\mathbf{v}^i = \Big(0, \dots, 0, \underbrace{\binom{n-1}{i-1}k^{i-1}}_{i\text{-component}}, 0, \dots, 0\Big), \quad 1 \leqq i \leqq n.$$

Theorem 2 shows that the polyhedron $\langle\mathcal{F}\rangle$ has a much more simpler form for $k>1$ than for $k=1$. Finally we note that E. K. R. families in other structures were considered by several authors (see, for instance, [4], [5], [6] and [9]).

2. Proof of Theorem 2

The vectors $\mathbf{v}^0, \dots, \mathbf{v}^n$ are elements of $\mathcal{F}$, i.e. E. K. R. vectors, since $\emptyset$ and the sets $\{\mathbf{x}\in N_i(P)\colon x_1=1\}$, $i\in\{1, \dots, n\}$, are E. K. R. families. From inequality (3) which we state and prove below we may derive that for all $\mathbf{f}\in\mathcal{F}$ it holds $f_i \leqq \binom{n-1}{i-1}k^{i-1}$, $i\in\{1, \dots, n\}$. Thus the vectors $\mathbf{v}^0, \dots, \mathbf{v}^n$ are not a convex combination of other E. K. R. vectors, hence they are vertices of $\langle\mathcal{F}\rangle$. Now we only must prove that each element of $\mathcal{F}$ (hence, each element of $\langle\mathcal{F}\rangle$) is a convex combination of $\mathbf{v}^0, \dots, \mathbf{v}^n$. Since $f_0=0$ for all $\mathbf{f}\in\mathcal{F}$ because of (2), this is valid, if

$$\sum_{i=1}^{n} \frac{f_i}{\binom{n-1}{i-1}k^{i-1}} \leqq 1 \quad \text{for all} \quad \mathbf{f}\in\mathcal{F}, \tag{3}$$

since then

$$\mathbf{f} = \left(1-\sum_{i=1}^{n} \frac{f_i}{\binom{n-1}{i-1}k^{i-1}}\right)\mathbf{v}^0 + \sum_{i=1}^{n} \frac{f_i}{\binom{n-1}{i-1}k^{i-1}}\mathbf{v}^i$$

is a convex combination of the vertices $\mathbf{v}^0, \dots, \mathbf{v}^n$. Thus, the proof of Theorem 2 reduces to the proof of the so-called LYM-inequality (3). We prove (3) by the method of twice counting (or better, twice summing). At first we must define what we are counting.

A subset C of P is called a chain iff for any $\mathbf{x}, \mathbf{y}\in C$ $\mathbf{x}\leqq\mathbf{y}$ or $\mathbf{x}\geqq\mathbf{y}$ holds. A maximal chain in P is a chain which consists of $n+1$ elements (i.e. it contains exactly one element of each level). Let $\mathfrak{C}$ be the set of all maximal chains in P. We will define an equivalence relation on $\mathfrak{C}$. For that let $\mathfrak{C}^*$ be the set of all pairs $(\pi, \mathbf{a})$, where π is a permutation of $\{1, \dots, n\}$ and $\mathbf{a}$ is an element of $N_n(P)$. Let $\iota\colon \mathfrak{C}\to\mathfrak{C}^*$ be that mapping, where, for $C=(\mathbf{c}^0<\dots<\mathbf{c}^n)$, it holds $\iota(C)=(\pi, \mathbf{a})$ iff $\{\pi(1)\}=Z(\mathbf{c}^{n-1})$, $\{\pi(1), \pi(2)\}=Z(\mathbf{c}^{n-2})$, $\dots$, $\{\pi(1), \dots, \pi(n)\}=Z(\mathbf{c}^0)$, and $\mathbf{a}=\mathbf{c}^n$.

Example 1. $n=4$, $k=2$, $C=((0, 0, 0, 0)<(0, 1, 0, 0)<(0, 1, 0, 2)<(0, 1, 1, 2)<$ $<(2, 1, 1, 2))$, $\iota(C)=\left(\begin{pmatrix}1&2&3&4\\1&3&4&2\end{pmatrix}, (2, 1, 1, 2)\right)$.
Obviously, ι is bijective.

Let $\zeta=\begin{pmatrix}1&2&\dots&n\\2&3&\dots&1\end{pmatrix}$. Now define the mapping $\varphi\colon \mathfrak{C}^*\to\mathfrak{C}^*$ by $\varphi((\pi, \mathbf{a}))=(\sigma, \mathbf{b})$, where $\sigma=\pi\circ\zeta$ (i.e. $\sigma(i)=\pi(\zeta(i))$ for all i) and

$$b_j = \begin{cases} a_j+1, & \text{if } j = \pi(1),\\ a_j, & \text{otherwise.}\end{cases}$$

Here and in all what follows the addition by the components of vectors is modulo k, and the congruence classes modulo k are $1, 2, \dots, k$.

Example 2. $n=4$, $k=2$, $\varphi\left(\left(\begin{pmatrix}1&2&3&4\\1&3&4&2\end{pmatrix}, (2, 1, 1, 2)\right)\right)=\left(\begin{pmatrix}1&2&3&4\\3&4&2&1\end{pmatrix}, (1, 1, 1, 2)\right)$.
Obviously, the inverse of φ is given by $\varphi^{-1}((\pi, \mathbf{a}))=(\varrho, \mathbf{c})$, where $\varrho=\pi\circ\zeta^{-1}$

$=\pi\circ\zeta^{kn-1}$ and

$$c_j = \begin{cases} a_j - 1, & \text{if } j = \pi(n), \\ a_j, & \text{otherwise.} \end{cases}$$

At last, it is easy to see that φ^{kn} is the identical mapping which we denote by φ^0. Now we may define an equivalence relation on $\mathfrak{C}$ by $C \sim C'$ iff $\iota^{-1}\circ\varphi^i\circ\iota(C) = C'$ for some i. Let $[C]$ be the equivalence class containing C. To each equivalence class $[C]$ we associate the set $\{C\}$ of elements of P which are contained in some chain being contained in $[C]$. Let $[\mathfrak{C}]$ be the set of all equivalence classes $[C]$. Now let F be an E. K. R. family in P and $\mathbf{f}$ be its E. K. R. vector, i.e. $f_i = |N_i(P) \cap F|$ for all i. We sum $1/r(\mathbf{x})$ over all pairs $(\mathbf{x}, [C])$, where $\mathbf{x} \in F \cap \{C\}$. ($F$ contains only elements of rank greater than 0 because of (2)). In Lemma 1 we will prove that for each element $\mathbf{x}$ of P there exist exactly $r(\mathbf{x})!(n-r(\mathbf{x}))!\,k^{n-r(\mathbf{x})}$ equivalence classes $[C]$ of chains such that $\mathbf{x} \in \{C\}$. Thus we obtain for our sum by fixing at first the first coordinate of the pairs the value

$$\sum_{i=1}^{n} \frac{f_i\, i!(n-i)!\,k^{n-i}}{i}.$$

In Lemma 2 we will show that each set $\{C\}$ contains at most l elements of F, if it contains one element of F with rank l. Thus we may estimate our sum by fixing at first the second coordinate of the pairs in the following way.

$$\sum_{[C]\in[\mathfrak{C}]} \sum_{\mathbf{x}\in\{C\}\cap F} \frac{1}{r(\mathbf{x})} \leqq \sum_{[C]\in[\mathfrak{C}]} \sum_{\mathbf{x}\in\{C\}\cap F} \frac{1}{\min\{r(\mathbf{x}):\ \mathbf{x}\in\{C\}\cap F\}}$$

$$\leqq \sum_{[C]\in[\mathfrak{C}]} \frac{\min\{r(\mathbf{x}):\ \mathbf{x}\in\{C\}\cap F\}}{\min\{r(\mathbf{x}):\ \mathbf{x}\in\{C\}\cap F\}} = \sum_{[C]\in[\mathfrak{C}]} 1 = \frac{k^n n!}{kn}.$$

(There are exactly $k^n n!$ maximal chains in P, and each equivalence class contains exactly kn chains.) Summarizing these results we obtain

$$\sum_{i=1}^{n} \frac{f_i\, i!(n-i)!\,k^{n-i}}{i} \leqq \frac{k^n n!}{kn}$$

and, equivalently,

$$\sum_{i=1}^{n} \frac{f_i}{\binom{n-1}{i-1} k^{i-1}} \leqq 1,$$

and we are done after proving Lemmas 1 and 2.

Lemma 1. *For each element* $\mathbf{x}$ *of* P *there are exactly* $r(\mathbf{x})!\,(n-r(\mathbf{x}))!\,k^{n-r(\mathbf{x})}$ *equivalence classes* $[C]$ *of chains such that* $\mathbf{x} \in \{C\}$.

Proof. Obviously, each element $\mathbf{x}$ of P is contained in exactly $r(\mathbf{x})!\,(n-r(\mathbf{x}))!\,k^{n-r(\mathbf{x})}$ maximal chains of P. Thus, it is sufficient to prove that any equivalent chains C and C' containing an arbitrary $\mathbf{x} \in P$ are equal. Let

(4) $$\iota^{-1}\circ\varphi^i\circ\iota(C) = C'.$$

Further let $\iota(C)=(\pi, \mathbf{a})$ and $\iota(C')=(\sigma, \mathbf{b})$. By definition of ι we have

$$\pi(\{1, \ldots, n-r(\mathbf{x})\}) = \sigma(\{1, \ldots, n-r(\mathbf{x})\}) = Z(\mathbf{x}), \tag{5}$$

$$a_j = b_j \quad \text{for all} \quad j \notin Z(\mathbf{x}). \tag{6}$$

Because of (4) and the definition of φ it is $\sigma=\pi\circ\zeta^i$. Thus it must be $i\equiv 0 \pmod n$ because of (5) and, furthermore, $i\equiv 0 \pmod{kn}$ because of (6). (If $r(\mathbf{x})=n$, one can derive $i\equiv 0 \pmod{kn}$ directly from (6)). But then $C=C'$. ∎

Lemma 2. *Let F be an E. K. R. family in P and $[C]$ an equivalence class of chains. Let $l=\min\{r(\mathbf{x})\colon \mathbf{x}\in F\cap\{C\}\}$. Then $\{C\}$ contains at most l elements of F.*

Proof. Assume that $|\{C\}\cap F|>l$. Let $\mathbf{x}^0\in\{C\}\cap F$ and $r(\mathbf{x}^0)=l$. Further, let $\mathbf{x}^0\in C^0\in[C]$ and $\iota(C^0)=(\pi, \mathbf{a})$. We define $C^i=\iota^{-1}\circ\varphi^i\circ\iota(C^0)$, $i\in\{0, \ldots, kn-1\}$. Obviously, $[C]=\{C^0, \ldots, C^{kn-1}\}$ and $\{C\}=C^0\cup\ldots\cup C^{kn-1}$. Let

$$A^j = C^{jn}\cup C^{jn+1}\cup\ldots\cup C^{jn+n-1}, \quad j\in\{0, \ldots, k-1\},$$

$$B^h = C^h\cup C^{h+n}\cup\ldots\cup C^{h+(k-1)n}, \quad h\in\{0, \ldots, n-1\}.$$

It holds that $\mathbf{x}\in A^j\cap B^h$ iff $\mathbf{x}\in C^{jn+h}$ since $C^i\cap C^{i'}=\emptyset$ if $0\leqq i<i'\leqq kn-1$ (see the proof of Lemma 1). By (1) and (2) we have

$$|B^h\cap F| \leqq 1 \quad \text{for all} \quad h\in\{0, \ldots, n-1\}. \tag{7}$$

In the following we will use the

Statement. *If $\mathbf{x}\in C^h\cap F$, $\mathbf{y}\in B^{h'}\cap F$ $(h, h'\in\{0, \ldots, n-1\})$ and $0<h'-h\leqq n-r(\mathbf{x})$, then $\mathbf{y}\in C^{h'}$ and $r(\mathbf{y})\leqq r(\mathbf{x})+h'-h-1$.*

Proof of the Statement. To prove that $\mathbf{y}\in C^{h'}$ we must show that $\mathbf{y}\in A^0$. Assume that $\mathbf{y}\in A^j$, i.e. $\mathbf{y}\in C^{jn+h'}$, where $j\in\{1, \ldots, k-1\}$. We will prove that there is no index i such that $x_i=y_i\in\{1, \ldots, k\}$ what is a contradiction to (2). By definition of ι and φ we have, for $1\leqq i\leqq h$, $x_{\pi(i)}=0$ or $x_{\pi(i)}=a_{\pi(i)}+1$ and $y_{\pi(i)}=0$ or $y_{\pi(i)}=a_{\pi(i)}+j+1$. Further, if $h<i\leqq h'$, it is $x_{\pi(i)}=0$ since $h'-h\leqq n-r(\mathbf{x})$ (note that $x_m=0$ iff $m\in\pi\circ\zeta^h(\{1, \ldots, n-r(\mathbf{x})\})$. At last, if $h'<i\leqq n$, we have $x_{\pi(i)}=0$ or $x_{\pi(i)}=a_{\pi(i)}$ and $y_{\pi(i)}=0$ or $y_{\pi(i)}=a_{\pi(i)}+j$. Consequently, $\mathbf{y}\in A^0$. Now assume that

$$r(\mathbf{y}) \geqq r(\mathbf{x})+h'-h. \tag{8}$$

We will show that $\mathbf{y}>\mathbf{x}$ what is a contradiction with (1). Because of the previous observations it is sufficient to prove that $Z(\mathbf{y})\subseteq Z(\mathbf{x})$. It is

$$x_m = 0 \quad \text{iff} \quad m\in\pi\circ\zeta^h(\{1, \ldots, n-r(\mathbf{x})\})$$

and

$$y_m = 0 \quad \text{iff} \quad m\in\pi\circ\zeta^{h'}(\{1, \ldots, n-r(\mathbf{y})\}).$$

We have $1+h<1+h'$ (supposition of the statement) and $n-r(\mathbf{x})+h\geqq n-r(\mathbf{y})+h'$ because of (8). Thus, $Z(\mathbf{y})\subseteq Z(\mathbf{x})$, and the proof of the statement is complete. ∎

We go on with the proof of Lemma 2. We may suppose $l<n$ since it follows from (7) that $|\{C\}\cap F|\leqq n$. Let $h_0=0$. We have $\mathbf{x}^{h_0}\in C^{h_0}\cap F$. If $B^h\cap F=\emptyset$ for $h=1, \dots, n-l$, then because of (7) $|\{C\}\cap F|\leqq l$ holds, a contradiction. Thus there are h_1 and $\mathbf{x}^1$ such that $h_0<h_1\leqq n-l$ and $\mathbf{x}_1\in B^{h_1}\cap F$. (We take the minimal possible value h_1, i.e. $B^h\cap F=\emptyset$ for $h_0<h<h_1$.) From the Statement we derive that $\mathbf{x}^1\in C^{h_1}$ and $r(\mathbf{x}^1)\leqq l+h_1-1$. Suppose we have already found $h_0, \dots, h_i$ and $\mathbf{x}^0, \dots, \mathbf{x}^i$ $(0\leqq i\leqq l-1)$ such that

$$(9) \qquad h_0<h_1<\dots<h_i\leqq n-l+i-1,$$

$$(10) \qquad \mathbf{x}^j\in C^{h_j}, \quad j\in\{0, \dots, i\},$$

$$(11) \qquad B^h\cap F=\emptyset \quad \text{for} \quad h_j<h<h_{j+1}, \quad j\in\{0, \dots, i-1\},$$

$$(12) \qquad r(\mathbf{x}^j)\leqq l+h_j-j, \quad j\in\{0, \dots, i\}.$$

If $B^h\cap F=\emptyset$ for $h=h_i+1, \dots, n-l+i$, then $|\{C\}\cap F|\leqq l$ because of (7), (10), and (11) (note that $|(B^0\cup\dots\cup B^{h_i})\cap F|=i+1$). This contradicts our assumption. Thus, there are h_{i+1} and $\mathbf{x}^{i+1}$ such that

$$(13) \qquad h_i<h_{i+1}\leqq n-l+i, \quad \mathbf{x}^{i+1}\in B^{h_{i+1}}\cap F$$

and

$$(14) \qquad B^h\cap F=\emptyset \quad \text{for} \quad h_i<h<h_{i+1}.$$

Because of (12) and (13) we have

$$h_i<h_{i+1}\leqq n-l+i\leqq n-\big(r(\mathbf{x}^i)-h_i+i\big)+i=n-r(\mathbf{x}^i)+h_i.$$

From the Statement (put $\mathbf{x}=\mathbf{x}^i$, $\mathbf{y}=\mathbf{x}^{i+1}$, $h=h_i$, and $h'=h_{i+1}$) and (12) we derive that

$$(15) \qquad \mathbf{x}^{i+1}\in C^{h_{i+1}}$$

and

$$(16) \quad r(\mathbf{x}^{i+1})\leqq r(\mathbf{x}^i)+h_{i+1}-h_i-1\leqq l+h_i-i+h_{i+1}-h_i-1=l+h_{i+1}-(i+1).$$

(13)—(16) imply that (9)—(12) hold for $i+1$ instead of i, and thus by induction. (9)—(12) hold for all $i\in\{0, \dots, l\}$. We conclude that there are h_l and $\mathbf{x}^l$ such that

$$(17) \qquad l\leqq h_l\leqq n-1 \quad \text{(by (9))},$$

$$(18) \qquad \mathbf{x}^l\in C^{h_l} \quad \text{(by (10))},$$

and

$$(19) \qquad r(\mathbf{x}^l)\leqq h_l \quad \text{(by (12))}.$$

We will show that $\mathbf{x}^0$ and $\mathbf{x}^l$ do not satisfy (2). Then all is done. For $1\leqq i\leqq n-l$ we have $x^0_{\pi(i)}=0$. If $n-l<i\leqq h_l$, it holds $x^0_{\pi(i)}=a_{\pi(i)}$ but $x^l_{\pi(i)}=a_{\pi(i)}+1$ by definition of ι and φ and because of (18). For $h_l<i\leqq n$ we have $x^l_{\pi(i)}=0$ since $Z(\mathbf{x}^l)=\pi\circ\zeta^{h_l}(\{1, \dots, n-r(\mathbf{x}^l)\})$ and $n-r(\mathbf{x}^l)+h_l\geqq n-h_l+h_l=n$ (because of (19)). ∎

3. A generalization

We can apply Theorems 1 and 2 also to the poset Q which is a product of n factors

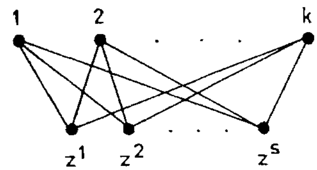

Fig. 2

i.e. Q consists of vectors $\mathbf{x}=(x_1, \ldots, x_n)$ with $x_i \in \{z^1, \ldots, z^s, 1, \ldots, k\}$ for all i, and the ordering is given by $\mathbf{x} \leqq \mathbf{y}$ iff $x_i = y_i$ or $x_i \in \{z^1, \ldots, z^s\}$ and $y_i \in \{1, \ldots, k\}$ for all i. Analogously to the introduction we define $Z(\mathbf{x}) = \{i\colon x_i \in \{z^1, \ldots, z^s\}\}$, $r(\mathbf{x}) = n - |Z(\mathbf{x})|$ and $N_i(Q) = \{\mathbf{x} \in Q\colon r(\mathbf{x}) = i\}$. An E. K. R. family G in Q is defined as in (1) and (2) and an E. K. R. vector $\mathbf{g} = (g_0, \ldots, g_n)$ is a vector for which there exists an E. K. R. family G in Q with $|N_i(Q) \cap G| = g_i$. Let $\mathscr{G}$ be the set of all E. K. R. vectors in Q. From the following Theorem 3 it follows that the convex closure of $\mathscr{G}$ and $\langle \mathscr{F} \rangle$ have the same vertices.

Theorem 3. $\mathscr{F} = \mathscr{G}$.

Proof. Let $\mathbf{f} \in \mathscr{F}$ and let F be an E. K. R. family with $|N_i(P) \cap F| = f_i$ for all i. Consider the function $\psi\colon P \to Q$ for which $\psi(\mathbf{x}) = \mathbf{y}$ iff

$$y_i = \begin{cases} x_i & \text{if } x_i \in \{1, \ldots, k\}, \\ z_1 & \text{if } x_i = 0. \end{cases}$$

Obviously, $G = \psi(F)$ is an E. K. R. family in Q, and $|N_i(Q) \cap G| = f_i$. Thus, $\mathbf{f} \in \mathscr{G}$.

Conversely, let $\mathbf{g} \in \mathscr{G}$ and let G be an E. K. R. family with $|N_i(Q) \cap G| = g_i$ for all i. Consider the function $\tau\colon Q \to P$ for which $\tau(\mathbf{x}) = \mathbf{y}$ iff

$$y_i = \begin{cases} x_i & \text{if } x_i \in \{1, \ldots, k\}, \\ 0 & \text{if } x_i \in \{z^1, \ldots, z^s\}. \end{cases}$$

τ restricted to G is injective, for suppose $\tau(\mathbf{x}) = \tau(\mathbf{w}) = \mathbf{y}$ for some $\mathbf{x}, \mathbf{w} \in G$. Then $x_i = w_i = y_i$ if $y_i \in \{1, \ldots, k\}$ and $x_i, w_i \in \{z^1, \ldots, z^s\}$ if $y_i = 0$ $(i \in \{1, \ldots, n\})$. But, if $x_i, w_i \in \{z^1, \ldots, z^s\}$ and $x_i \neq w_i$, then condition (2) is violated. Hence, $x_i = w_i$ for all i, i.e. $\mathbf{x} = \mathbf{w}$. Further it is easy to see that $F = \tau(G)$ is an E. K. R. family in P and $|N_i(P) \cap F| = g_i$ for all i. Consequently, $\mathbf{g} \in \mathscr{F}$. ▮

References

[1] K. ENGEL, *Maximale h-Familien in endlichen Ordnungen, Hansel-Ordnungen und monotone Funktionen*, Dissertation A, Wilh.-Pieck-Univ., Rostock, 1981.

[2] P. ERDŐS, CHAO KO and R. RADO, Intersection theorems for systems of finite sets, *Quart. J. Math. Oxford* **12** (1961), 313—320.

[3] P. L. ERDŐS, P. FRANKL and G. O. H. KATONA, Intersecting Sperner families and their convex hulls. *Combinatorica* **4** (1984), 21—34.

[4] C. Greene, G. O. H. Katona and D. J. Kleitman, Extensions of the Erdős—Ko—Rado Theorem, *Studies in Appl. Math.* **55** (1976), 1—8.
[5] H.-D. O. F. Gronau, *Zur Theorie der extremalen Familien von Teilmengen einer endlichen Menge*, Dissertation B, Wilh.-Pieck-Univ., Rostock, 1982.
[6] W. N. Hsieh, Families of intersecting finite vector spaces, *J. Combin. Theory Ser. A* **18** (1975), 252—261.
[7] K. Leeb, Salami-Taktik beim Quader-Packen, *Arbeitsberichte des Instituts für Mathematische Maschinen und Datenverarbeitung (Informatik)*, Friedrich Alexander Universität Erlangen Nürnberg **11**, Nr. 5 (1978), 1—15.
[8] N. Metropolis and G.-C. Rota, Combinatorial structure of the faces of the n-cube, *SIAM J. Appl. Math.* **35** (1978), 689—694.
[9] A. Moon, An analogue of the Erdős—Ko—Rado Theorem for the Hamming Schemes $H(n, q)$, *J. Combin. Theory Ser. A* **32** (1982), 386—390.

Konrad Engel

Sektion Mathematik, Wilhelm-Pieck-Universität,
2500 Rostock, German Democratic Republic